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A quantitative asymptotic formula for a general class of discrete operators

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ABSTRACT

Here we give some quantitative versions of the Voronovskaja formula for a general class of discrete operators, not necessarily positive. Applications to various generalizations of the Szász–Mirak'jan operator and a Jackson type sampling operator are given.

1. Introduction

In paper [1] we studied the pointwise approximation properties for a general class of discrete, not necessarily positive, operators defined by

$$(S_n f)(t) = \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}) f(\nu_{n,k}), \quad n \in \mathbb{N},\ t \in I,$$

where I is a fixed interval (bounded or not) in $\mathbb{R}$ and, for every fixed $n \in \mathbb{N}$, $(\nu_{n,k})_{k\in\mathbb{N}_0} \subset I$ is a sequence satisfying suitable assumptions. Here $K_n(t, \nu_{n,k})$ is a kernel function satisfying suitable assumptions and the function f belongs to the domain of the operator S_n.

This class of operators contains various classical discrete operators: the Bernstein polynomials and many of their generalizations (see for example [2–10], for a survey on these operators see also [1]) and the generalized sampling series introduced by Butzer and his school in Aachen (see e.g. [11,12]) which have important applications in signal processing.

So this general class enables us to give a unitary approach to the study of the convergence properties for a large class of discrete operators.

In this paper we study some quantitative versions of the Voronovskaja formula for the operator S_n in terms of the least concave majorant $\widetilde{\omega}$ of the classical modulus of continuity for continuous functions (see for example [2,13,14]).

We use a technique developed in [15] by Gonska, Pitul and Rasa which is based on the Peetre K-functional (see [16,17]) and on a new estimate of the remainder in the Taylor formula. The use of K-functionals is very common in many areas of mathematical analysis in particular in approximation theory and in interpolation theory (see for example [18–20,13,21,14]).

In Section 2 we give the general Voronovskaja formula for the operator S_n and we state some quantitative versions which give the uniform convergence on bounded subsets of I.

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In Section 3 we apply the results to some generalizations of the Szász–Mirak'jan operator. One was introduced in [1] where the generating function is of type $\varphi(t) = p(t)e^t$, and $p(t)$ is a polynomial of type $p(t) = t^r + b_1 t^{r-1} + \cdots + b_r$ with non-negative coefficients, $t \in I = [0, +\infty[$. Another one is constructed starting from a modification studied in [22].

In Section 4 we consider a King type version of the Szász–Mirak'jan operator studied in various papers (see for example [22]). The so-called King type operators preserve the function $e_2(t) = t^2$ and provide in general a better order of approximation (see [7,23,9] for Bernstein type operators).

The last section is devoted to the study of a Jackson type generalized sampling operator generated by a Jackson type kernel (see e.g. [24,25]). In particular this example gives a new contribution to the theory of the generalized sampling series.

2. A general Voronovskaja type theorem

In the following we will denote by I a fixed interval (bounded or not) in $\mathbb{R}$ and, for every fixed $n \in \mathbb{N}$, by $\Gamma_n = (\nu_{n,k})_{k\in\mathbb{N}_0} \subset I$ a sequence such that

$$0 < \nu_{n,k+1} - \nu_{n,k} =: \lambda_{n,k} \leq \lambda_n,$$

where λ_n are positive real numbers and $\lim_{n\to+\infty} \lambda_n = 0$.

In [1] we introduced a sequence $S = (S_n)$ of operators of the form

$$(S_n f)(t) = \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}) f(\nu_{n,k}), \quad n \in \mathbb{N},\ t \in I, \tag{1}$$

for $f \in \mathrm{Dom}\, S := \bigcap_{n\in\mathbb{N}} \mathrm{Dom}\, S_n$, where $\mathrm{Dom}\, S_n$ is the set of all functions $f : I \to \mathbb{R}$ for which (1) is well defined.

We put for $j \in \mathbb{N}_0$

$$m_j(K_n, t) := \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k})(\nu_{n,k} - t)^j$$

and

$$M_j(K_n, t) := \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})|\, |\nu_{n,k} - t|^j.$$

The family of functions $(K_n)_{n\in\mathbb{N}}$, $K_n : I \times \Gamma_n \to \mathbb{R}$ satisfies the following assumptions for every $t \in I$

(1) $m_0(K_n, t) = \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k}) = 1$, for every $n \in \mathbb{N}$,

(2) for every $n \in \mathbb{N}$ and $j = 1, 2$

$$-\infty < m_j(K_n, t) < +\infty$$

and there are $\alpha > 0$ and real numbers $\ell_j(t)$ such that

$$\lim_{n\to+\infty} n^\alpha m_j(K_n, t) = \ell_j(t), \quad j = 1, 2,$$

(3) for the above $\alpha > 0$, there is a positive constant $H(t)$ and $\overline{n} \in \mathbb{N}$ such that

$$n^\alpha M_2(K_n, t) \leq H(t)$$

for every $n \geq \overline{n}$ and, for every $\delta > 0$,

$$\sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})|(\nu_{n,k} - t)^2 = o(n^{-\alpha}) \quad (n \to +\infty).$$

Remark 1. Note that assumption (3) implies that

$$\sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})|\, |\nu_{n,k} - t|^j = o(n^{-\alpha})$$

for $j = 0, 1$.

Remark 2. If we assume that there is a positive constant a_n such that

$$a_n \leq \nu_{n,k+1} - \nu_{n,k}, \quad n \in \mathbb{N},$$

then $M_2(K_n, t) < +\infty$ implies $M_j(K_n, t) < +\infty$ for $j = 0, 1$. Indeed for example

$$\sum_{k=0}^{\infty} |K_n(t, \nu_{n,k})| = \left(\sum_{|\nu_{n,k}-t|<1} + \sum_{|\nu_{n,k}-t|\geq 1} \right) |K_n(t, \nu_{n,k})| := I_1 + I_2.$$

Then I_1 has only a finite number of terms while for I_2, using assumption (3), we get

$$I_2 \leq \sum_{|\nu_{n,k}-t|\geq 1} |K_n(t, \nu_{n,k})|(\nu_{n,k} - t)^2 \leq M_2(K_n, t) < +\infty.$$

In this case we have also that $L^\infty(I) \subset \mathrm{Dom}\, S$.

Remark 3. If (K_n) is non-negative then $L^\infty(I) \subset \mathrm{Dom}\, S$ is a direct consequence of assumption (1).

In [1] we proved the following general result which we restate with the proof for the sake of completeness.

Theorem 1. *Let $f \in \mathrm{Dom}\, S \bigcap L^\infty(I)$ be a function such that $f''(t)$ exists at a point $t \in I$. Under the above assumptions there holds*

$$\lim_{n\to+\infty} n^\alpha[(S_nf)(t) - f(t)] = f'(t)\ell_1(t) + \frac{f''(t)}{2}\ell_2(t).$$

Proof. Using a local Taylor's formula for the function f, there exists a bounded function h such that $\lim_{y\to 0} h(y) = 0$ and

$$f(\nu_{n,k}) = f(t) + f'(t)(\nu_{n,k} - t) + \frac{f''(t)}{2}(\nu_{n,k} - t)^2 + h(\nu_{n,k} - t)(\nu_{n,k} - t)^2.$$

Thus we have

$$\begin{aligned} n^\alpha[(S_nf)(t) - f(t)] &= n^\alpha f'(t)\sum_{k=0}^{+\infty} K_n(t, \nu_{n,k})(\nu_{n,k} - t) + n^\alpha \frac{f''(t)}{2}\sum_{k=0}^{+\infty} K_n(t, \nu_{n,k})(\nu_{n,k} - t)^2 \\ &\quad + n^\alpha \sum_{k=0}^{+\infty} K_n(t, \nu_{n,k})h(\nu_{n,k} - t)(\nu_{n,k} - t)^2 \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We immediately get

$$I_1 = n^\alpha f'(t)m_1(K_n, t), \qquad I_2 = n^\alpha \frac{f''(t)}{2} m_2(K_n, t).$$

Now we estimate I_3. Let $\varepsilon > 0$ be fixed. There exists $\delta > 0$ such that $|h(y)| \leq \varepsilon$ for every $|y| \leq \delta$. Hence

$$|I_3| \leq n^\alpha \sum_{|\nu_{n,k}-t|<\delta} |K_n(t, \nu_{n,k})h(\nu_{n,k} - t)|(\nu_{n,k} - t)^2 + n^\alpha \sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})h(\nu_{n,k} - t)|(\nu_{n,k} - t)^2 = I_3' + I_3''.$$

We obtain, by assumption 3,

$$|I_3'| \leq \varepsilon H(t)$$

for sufficiently large n. Moreover, choosing a constant $M > 0$ such that $|h(y)| \leq M$ we have

$$|I_3''| \leq Mn^\alpha \sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})|(\nu_{n,k} - t)^2 = o(1)$$

for $n \to +\infty$. Thus,

$$\lim_{n\to+\infty} |I_3| = 0$$

and so, using condition (2), the assertion follows. □

Remark 4. As remarked in [1], if I is an unbounded interval, we can relax the boundedness assumption on f assuming that there are two positive constants a, b such that

$$|f(t)| \leq a + bt^2, \quad \text{for every } t \in I.$$

The above result can be restated also for functions for which only f' exists at a point $t \in I$. In this case we have to assume that condition (1) holds and conditions (2) and (3) are replaced by

(2′) for every $t \in I$, $n \in \mathbb{N}$

$$-\infty < m_1(K_n, t) < +\infty$$

and there are $\alpha > 0$ and a real number $\ell_1(t)$ such that

$$\lim_{n\to+\infty} n^\alpha m_1(K_n, t) = \ell_1(t),$$

(3′) for the above $\alpha > 0$ and $t \in I$, there is a positive constant $H(t)$ and $\overline{n} \in \mathbb{N}$ such that

$$n^{\alpha} M_1(K_n, t) \leq H(t)$$

for every $n \geq \overline{n}$ and, for every $\delta > 0$, $t \in I$

$$\sum_{|\nu_{n,k}-t|\geq\delta} |K_n(t, \nu_{n,k})|\,|\nu_{n,k} - t| = o(n^{-\alpha}) \quad (n \to +\infty).$$

Theorem 2. *Let $f \in \operatorname{Dom} S \bigcap L^{\infty}(I)$ be a function such that $f'(t)$ exists at a point $t \in I$. Under the above assumptions there holds*

$$\lim_{n\to+\infty} n^{\alpha}[(S_n f)(t) - f(t)] = f'(t)\ell_1(t).$$

Proof. The proof is the same as before using the differentiability of the function f. □

Now our aim is to determine the order of the convergence in Theorem 1 using a suitable modulus of continuity. Let us denote by $C^0 = C^0(I)$ the space of all uniformly continuous and bounded functions $f : I \to \mathbb{R}$, provided with the usual supnorm $\|f\|_{\infty}$ and for $k \geq 1$ by $C^k = C^k(I)$ the subspace of C^0 whose elements f are k-times continuously differentiable and $f^{(k)} \in C^0$. For a given $\varepsilon > 0$ we define

$$\omega(f, \varepsilon) := \sup_{|x-y|<\varepsilon} |f(x) - f(y)|.$$

In [15], for $f \in C^m$, the following version of the Taylor formula is stated

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_m(f; x_0, x),$$

for $x_0, x \in I$, $m \geq 1$ and the remainder $R_m(f; x_0, x)$ is estimated by

$$|R_m(f; x_0, x)| \leq \frac{|x - x_0|^m}{m!}\omega(f^{(m)}; |x - x_0|).$$

Note that, since $f^{(m)}$ is continuous, there holds

$$\omega(f^{(m)}; |x - x_0|) = o(1)$$

for $x \to x_0$.

In what follows, we will need the following K-functional, introduced by Peetre ([16], see also [17]) and defined by

$$\mathcal{K}(\varepsilon, f) \equiv \mathcal{K}(\varepsilon, f, C^0, C^1) := \inf\{\|f - g\|_{\infty} + \varepsilon\|g'\|_{\infty} : g \in C^1\}$$

for $f \in C^0$ and $\varepsilon \geq 0$. In order to relate the K-functional to a modulus of continuity, we will quote the following lemma (see [17, Corollary 2.1] and [26, Lemma 12.1]).

Lemma 1. *For every $f \in C^0$ there holds*

$$\mathcal{K}(\varepsilon/2, f) = \frac{1}{2}\widetilde{\omega}(f, \varepsilon), \quad \varepsilon \geq 0.$$

Here $\widetilde{\omega}(f, \cdot)$ denotes the least concave majorant of $\omega(f, \cdot)$ (see e.g. [2,14]).

As in [15], we have the following estimate of the remainder $R_m(f; x_0, x)$ in terms of $\widetilde{\omega}$.

Lemma 2. *For $m \in \mathbb{N}_0$, let $f \in C^m$ and $x, x_0 \in I$. Then we have*

$$|R_m(f; x_0, x)| \leq \frac{|x - x_0|^m}{m!}\widetilde{\omega}\left(f^{(m)}, \frac{|x - x_0|}{m + 1}\right).$$

We study an estimate of the convergence in Theorem 1 in terms of the modulus $\widetilde{\omega}$, in case $m = 2$. In order to do that, we will consider the absolute moment

$$M_3(K_n, t) = \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})|\,|\nu_{n,k} - t|^3,$$

for every $t \in I$. We have the following

Theorem 3. *Let $f \in C^2$ be fixed and let $t \in I$. Under the assumptions of Theorem 1, if moreover $M_3(K_n, t) < +\infty$, there holds*

$$\left| n^\alpha [(S_n f)(t) - f(t)] - \ell_1(t) f'(t) - \ell_2(t) \frac{f''(t)}{2} \right| \leq |f'(t)| \, |n^\alpha m_1(K_n, t) - \ell_1(t)| + \frac{|f''(t)|}{2} |n^\alpha m_2(K_n, t) - \ell_2(t)| + n^\alpha \frac{M_2(K_n, t)}{2} \widetilde{\omega} \left(f'', \frac{1}{3} \frac{M_3(K_n, t)}{M_2(K_n, t)} \right).$$

Proof. Using the Taylor formula as in Theorem 1, we get

$$\left| [(S_n f)(t) - f(t)] - \frac{1}{n^\alpha} \left(\ell_1(t) f'(t) + \ell_2(t) \frac{f''(t)}{2} \right) \right| \leq |f'(t)| \left| m_1(K_n, t) - \frac{\ell_1(t)}{n^\alpha} \right| + \frac{|f''(t)|}{2} \left| m_2(K_n, t) - \frac{\ell_2(t)}{n^\alpha} \right| + \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| \, |h(\nu_{n,k} - t)| (\nu_{n,k} - t)^2 = I_1 + I_2 + I_3.$$

We evaluate only I_3. Putting $R_2(f; t, \nu_{n,k}) = h(\nu_{n,k} - t)(\nu_{n,k} - t)^2$, taking into account Lemma 2 with $m = 2$, we have

$$|I_3| \leq \frac{1}{2} \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| (\nu_{n,k} - t)^2 \widetilde{\omega} \left(f'', \frac{|\nu_{n,k} - t|}{3} \right) = \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| (\nu_{n,k} - t)^2 \mathcal{K} \left(\frac{|\nu_{n,k} - t|}{6}, f'' \right).$$

Let $g \in C^3$ be fixed. Then

$$\begin{aligned} |I_3| &\leq \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| (\nu_{n,k} - t)^2 \left[\|(f-g)''\|_\infty + \frac{|\nu_{n,k} - t|}{6} \|g'''\|_\infty \right] \\ &= \sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| (\nu_{n,k} - t)^2 \left[\|(f-g)''\|_\infty + \frac{\|g'''\|_\infty}{6} \frac{\sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| \, |\nu_{n,k} - t|^3}{\sum_{k=0}^{+\infty} |K_n(t, \nu_{n,k})| (\nu_{n,k} - t)^2} \right]. \end{aligned}$$

Thus taking the infimum over $g \in C^3$ we get

$$|I_3| \leq M_2(K_n, t) \mathcal{K} \left(\frac{1}{6} \frac{M_3(K_n, t)}{M_2(K_n, t)}, f'' \right).$$

So the assertion follows. □

Analogously we can obtain the following quantitative version of Theorem 2, using Lemma 2 for $m = 1$.

Theorem 4. *Let $f \in C^1$ be fixed and let $t \in I$. Under the assumptions of Theorem 2, if $M_2(K_n, t) < +\infty$, there holds*

$$|n^\alpha [(S_n f)(t) - f(t)] - \ell_1(t) f'(t)| \leq |f'(t)| \, |n^\alpha m_1(K_n, t) - \ell_1(t)| + n^\alpha M_1(K_n, t) \widetilde{\omega} \left(f', \frac{1}{2} \frac{M_2(K_n, t)}{M_1(K_n, t)} \right).$$

3. Szász–Mirak'jan type operators

In this section we apply the previous theory to some generalizations of the Szász–Mirak'jan operator. The first one was introduced in [1] and it is generated by functions of type $\varphi(t) = p(t) e^t$, where $p(t)$ is a positive polynomial of type $p(t) = t^r + b_1 t^{r-1} + \cdots + b_r$ with non-negative coefficients, $t \in I = [0, +\infty[$. The second one deals with a modification of the classical Szász–Mirak'jan operator studied in [22] in which we choose different samples.

Example 1. We use a set of sample values of type $k/(n+\gamma)$, where γ is a non-negative constant, i.e. $\nu_{n,k} = \frac{k}{n+\gamma}$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$. We define the operator

$$(T_n f)(t) = \frac{1}{\varphi(nt)} \sum_{k=0}^{+\infty} a_k (nt)^k f\left(\frac{k}{n+\gamma} \right)$$

where the coefficients a_k are given (uniquely) by the Taylor expansion

$$\varphi(t) = \sum_{k=0}^{+\infty} a_k t^k.$$

Note that in this case the domain of the operator T_n contains very large classes of functions; for example we can consider also functions with exponential growth (for the Szász–Mirak'jan operator see e.g. [27,3,4]). In what follows, we will assume for the sake of simplicity that $f \in L^\infty(I)$. Since the kernel

$$K_n\left(t, \frac{k}{n+\gamma}\right) = \frac{1}{\varphi(nt)} a_k (nt)^k$$

is a positive kernel, $L^\infty(I) \subset \operatorname{Dom} T = \bigcap_{n\in\mathbb{N}} \operatorname{Dom} T_n$.

Assumption (1) is obviously satisfied. Moreover in [1], we calculated the first four moments of K_n:

$$m_1(K_n, t) = \frac{nt}{n+\gamma}\frac{\varphi'(nt)}{\varphi(nt)} - t.$$

$$m_2(K_n, t) = M_2(K_n, t) = \frac{nt}{n+\gamma}\left(\frac{\varphi'(nt)}{\varphi(nt)}(\frac{1}{n+\gamma} - 2t) + \frac{\varphi''(nt)}{\varphi(nt)}\frac{nt}{n+\gamma}\right) + t^2.$$

$$\begin{aligned} m_3(K_n, t) \;=\; & \frac{1}{\varphi(nt)}\frac{nt}{(n+\gamma)^3}(\varphi'(nt) + 3nt\varphi''(nt) + n^2t^2\varphi'''(nt)) - \frac{3t}{\varphi(nt)}\frac{nt}{(n+\gamma)^2}(\varphi'(nt) \\ & + nt\varphi''(nt)) + \frac{3t^2}{\varphi(nt)}\frac{nt}{(n+\gamma)}\varphi'(nt) - t^3. \end{aligned}$$

$$\begin{aligned} m_4(K_n, t) \;=\; & M_4(K_n, t) = \frac{1}{\varphi(nt)}\frac{nt}{(n+\gamma)^4}(\varphi'(nt) + 7nt\varphi''(nt) + 6n^2t^2\varphi'''(nt) + n^3t^3\varphi^{iv}(nt)) \\ & - \frac{4t}{\varphi(nt)}\frac{nt}{(n+\gamma)^3}(\varphi'(nt) + 3nt\varphi''(nt) + n^2t^2\varphi'''(nt)) \\ & + \frac{6t^2}{\varphi(nt)}\frac{nt}{(n+\gamma)^2}(\varphi'(nt) + nt\varphi''(nt)) - \frac{4t^3}{\varphi(nt)}\frac{nt}{(n+\gamma)}\varphi'(nt) + t^4. \end{aligned}$$

As a consequence we obtain, for $t > 0$,

$$\lim_{n\to+\infty} nm_1(K_n, t) = r - \gamma t, \qquad \lim_{n\to+\infty} nm_2(K_n, t) = t,$$

while for $t = 0$ all the moments are null and so $\ell_1(t) = \ell_2(t) = 0$. This means that condition (2) holds with $\alpha = 1$. For what concerns condition (3), in [1] we proved that

$$\lim_{n\to+\infty} nm_4(K_n, t) = 0$$

and for $\delta > 0$,

$$\sum_{|(k/n+\gamma)-t|\geq\delta} \left|K_n\left(t, \frac{k}{n+\gamma}\right)\right| \left(\frac{k}{n+\gamma} - t\right)^2 \leq \frac{1}{\delta^2} m_4(K_n, t) = o(n^{-1})$$

for $n \to +\infty$. So we obtain the following Voronovskaja formula

$$\lim_{n\to+\infty} n[(T_nf)(t) - f(t)] = (r - \gamma t)f'(t) + t\frac{f''(t)}{2},$$

at every point $t > 0$ in which $f''(t)$ exists.

Note that in case $r = 0$ and $\gamma = 0$, T_n is the classical Szász–Mirak'jan operator:

$$(T_nf)(t) = \sum_{k=0}^{\infty} e^{-nt}\frac{(nt)^k}{k!} f\left(\frac{k}{n}\right), \quad t \geq 0.$$

The asymptotic formula reduces to the classical one:

$$\lim_{n\to+\infty} n[(T_nf)(t) - f(t)] = t\frac{f''(t)}{2},$$

when $f''(t)$ exists. Other interesting generalizations were given in [5,6]. In order to obtain the quantitative version of the previous Voronovskaja formula, we will apply Theorem 3. We need the following

Lemma 3. *For the kernel K_n, for $t \in I$, we have $M_3(K_n, t) < +\infty$ and*

$$\frac{M_3(K_n, t)}{M_2(K_n, t)} = \mathcal{O}(n^{-1/2}), \quad n \to +\infty.$$

Moreover

$$|nm_1(K_n, t) - (r - \gamma t)| = \mathcal{O}(n^{-1}), \qquad |nm_2(K_n, t) - t| = \mathcal{O}(n^{-1}), \quad n \to +\infty.$$

Proof. Firstly as proved in [1] $m_4(K_n, t)$ is finite. Moreover using the Hölder–Schwarz inequality we have

$$\begin{aligned} M_3(K_n,t) &\le \left(\frac{1}{\varphi(nt)}\sum_{k=0}^{+\infty}\left(\frac{k}{n+\gamma}-t\right)^2 a_k(nt)^k\right)^{1/2}\left(\frac{1}{\varphi(nt)}\sum_{k=0}^{+\infty}\left(\frac{k}{n+\gamma}-t\right)^4 a_k(nt)^k\right)^{1/2} \\ &= \sqrt{m_2(K_n,t)}\sqrt{m_4(K_n,t)}. \end{aligned}$$

Hence for $t > 0$

$$\frac{M_3(K_n,t)}{M_2(K_n,t)} \le \sqrt{\frac{m_4(K_n,t)}{m_2(K_n,t)}}.$$

Now using the expression of the fourth moment $m_4(K_n, t)$ and the calculations given in Lemma 2 in [1] we get

$$\lim_{n\to+\infty} n^2 m_4(K_n,t) = 3t^2.$$

So

$$\frac{M_3(K_n,t)}{M_2(K_n,t)} = \mathcal{O}(n^{-1/2}), \quad n\to+\infty.$$

The second part follows by elementary calculations. □

So we have the following corollary for the modified Szász–Mirak'jan operator.

Corollary 1. *Let $f \in C^2$ be fixed and let $t > 0$. Then there holds*

$$\left| n[(T_nf)(t)-f(t)] - (r-\gamma t)f'(t) - t\frac{f''(t)}{2}\right| \le |f'(t)|\mathcal{O}(n^{-1}) + \frac{|f''(t)|}{2}\mathcal{O}(n^{-1}) + \mathcal{O}(1)\widetilde{\omega}\left(f'', \mathcal{O}(n^{-1/2})\right).$$

Example 2. As in the previous example we use here the same set of sample values $\nu_{n,k} = \frac{k}{n+\gamma}$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $\gamma \ge 0$. We define the operator (see [22])

$$(T_nf)(t) = \mathrm{e}^{-nu_n(t)}\sum_{k=0}^{+\infty}\frac{(nu_n(t))^k}{k!} f\left(\frac{k}{n+\gamma}\right)$$

with $t \ge 0$ and where

$$u_n(t) = \frac{\sqrt{1+4n^2t^2}-1}{2n}.$$

Here

$$K_n\left(t, \frac{k}{n+\gamma}\right) = \mathrm{e}^{-nu_n(t)}\frac{(nu_n(t))^k}{k!}$$

so that K_n is a positive kernel and assumption (1) is obviously satisfied. As before we will assume that $f \in L^\infty(I) \subset \mathrm{Dom}\, T$.

By elementary and tedious calculations we obtain the following expression for the first four moments of K_n:

$$m_1(K_n,t) = \frac{nu_n(t)}{n+\gamma} - t$$

$$m_2(K_n,t) = t^2 - \frac{2tnu_n(t)}{n+\gamma} + \frac{n(u_n(t)+nu_n^2(t))}{(n+\gamma)^2}$$

$$\begin{aligned} m_3(K_n,t) &= -t^3\left(1+\frac{3n^2}{(n+\gamma)^2}\right) + \frac{3}{2}t^2\left(\frac{n^2}{(n+\gamma)^3} - \frac{1}{n+\gamma}\right) + \frac{1}{2(n+\gamma)^3} \\ &\quad + \sqrt{1+4n^2t^2}\left(\frac{n^2t^2}{2(n+\gamma)^3} + \frac{3t^2}{2(n+\gamma)} - \frac{1}{2(n+\gamma)^3}\right) \end{aligned}$$

$$\begin{aligned} m_4(K_n,t) &= t^4\left(1+\frac{n^4}{(n+\gamma)^4}+\frac{6n^2}{(n+\gamma)^2}\right) + t^3\left(\frac{2}{(n+\gamma)} - \frac{6n^2}{(n+\gamma)^3}\right) - \frac{2t}{(n+\gamma)^3} + \frac{1}{2(n+\gamma)^4} \\ &\quad + \sqrt{1+4n^2t^2}\left(\frac{2n^2t^2}{(n+\gamma)^4} + \frac{2t}{(n+\gamma)^3} - \frac{2n^2t^3}{(n+\gamma)^3} - \frac{1}{2(n+\gamma)^4} - \frac{2t^3}{n+\gamma}\right). \end{aligned}$$

So we obtain, for $t > 0$,

$$\lim_{n\to+\infty} nm_1(K_n,t) = -\frac{1}{2} - \gamma t, \qquad \lim_{n\to+\infty} nm_2(K_n,t) = t,$$

while for $t = 0$ the moments are null, so $\ell_1(t) = \ell_2(t) = 0$. This means that condition (2) holds with $\alpha = 1$.

Moreover, since

$$\lim_{n\to+\infty} n^2 m_4(K_n,t) = 3t^2,$$

condition (3) is also satisfied. Thus we obtain the following Voronovskaja formula

$$\lim_{n\to+\infty} n[(T_nf)(t) - f(t)] = \left(-\frac{1}{2} - \gamma t\right) f'(t) + t\frac{f''(t)}{2},$$

at every point $t > 0$ in which $f''(t)$ exists.

In order to obtain the corresponding quantitative version we will apply Theorem 3. We need the following

Lemma 4. *For the kernel K_n, for $t \in I$, we have $M_3(K_n,t) < +\infty$ and*

$$\frac{M_3(K_n,t)}{M_2(K_n,t)} = \mathcal{O}(n^{-1/2}), \quad n \to +\infty.$$

Moreover

$$\left| nm_1(K_n,t) + \frac{1}{2} + \gamma t \right| = \mathcal{O}(n^{-1}), \qquad |nm_2(K_n,t) - t| = \mathcal{O}(n^{-1}), \quad n \to +\infty.$$

Proof. The proof is carried out as in Lemma 3. □

So we have the following

Corollary 2. *Let $f \in C^2$ be fixed and let $t > 0$. Then there holds*

$$\left| n[(T_nf)(t) - f(t)] + \left(\frac{1}{2} + \gamma t\right) f'(t) - t\frac{f''(t)}{2} \right| \le |f'(t)|\mathcal{O}(n^{-1}) + \frac{|f''(t)|}{2}\mathcal{O}(n^{-1}) + \mathcal{O}(1)\widetilde{\omega}\left(f'', \mathcal{O}(n^{-1/2})\right).$$

Note that in the above examples, as a consequence of Corollaries 1 and 2, the convergence in the Voronovskaja formulae is uniform on every compact interval in $]0, +\infty[$.

4. Some kinds of King type operators

In [7] families of positive linear operators which preserve the function $e_2(t) = t^2$ were introduced. These kinds of operators in general provide a better error estimate. Thus we consider now a sequence L_n of linear operators of the general form

$$(L_nf)(t) = \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k}) f(\nu_{n,k}), \quad n \in \mathbb{N},\ t \in I = \mathbb{R}_0^+,$$

such that $(L_ne_0)(t) = e_0(t)$ and $(L_ne_2)(t) = e_2(t)$, where $e_0(t) = 1$ and $e_2(t) = t^2$ for every $t \in \mathbb{R}_0^+$. In this case we have

$$\begin{aligned} m_2(K_n,t) = \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})(\nu_{n,k} - t)^2 &= \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})\nu_{n,k}^2 + \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})t^2 - 2t\sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})\nu_{n,k} \\ &= 2t^2 - 2t\sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})\nu_{n,k} = 2t\left(t - \sum_{k=0}^{+\infty} K_n(t,\nu_{n,k})\nu_{n,k}\right) \\ &= -2tm_1(K_n,t). \end{aligned}$$

So if we consider King type operators, condition (1) holds and for condition (2) we have that if there are $\alpha > 0$ and a real number $\ell_1(t)$ such that

$$\lim_{n\to+\infty} n^\alpha m_1(K_n,t) = \ell_1(t),$$

then

$$\lim_{n\to+\infty} n^\alpha m_2(K_n,t) = -2t\ell_1(t).$$

Note that if the kernel K_n is positive then the first condition in (3) is satisfied. Indeed, for example,

$$n^\alpha m_2(K_n,t) = -2tn^\alpha m_1(K_n,t) \le 2t(1 - \ell_1(t)) = H(t).$$

As a consequence, we obtain the following general result for King type operators

Corollary 3. *Let (L_n) be a sequence of positive linear operators of King type. Assume that there exists $\alpha > 0$ with*

$$\lim_{n\to+\infty} n^{\alpha} m_1(K_n, t) = \ell_1(t), \quad t \in I$$

and that the second condition in (3) is satisfied for every $\delta > 0$. Then for $f \in L^{\infty}(I)$ such that $f''(t)$ exists at a point $t \in I$ there holds

$$\lim_{n\to+\infty} n^{\alpha}[(L_nf)(t) - f(t)] = (f'(t) - tf''(t))\ell_1(t).$$

As a special case, we consider now the operator given in [22] and defined by

$$(L_n^*f)(t) = \sum_{k=0}^{+\infty} K_n^*(t, \nu_{n,k}) f(\nu_{n,k}) = \mathrm{e}^{-nu_n(t)} \sum_{k=0}^{+\infty} \frac{(nu_n(t))^k}{k!} f\left(\frac{k}{n}\right)$$

with

$$u_n(t) = \frac{-1+\sqrt{4n^2t^2+1}}{2n}, \quad t \ge 0,\ n \in \mathbb{N}.$$

In [22] it was proved that

$$m_1(K_n^*, t) = -t - \frac{1}{2n} + \frac{\sqrt{4n^2t^2+1}}{2n},$$
$$m_2(K_n^*, t) = 2t^2 + \frac{t}{n} - \frac{t\sqrt{4n^2t^2+1}}{n},$$
$$m_3(K_n^*, t) = -4t^3 + \frac{1}{2n^3} + \left(\frac{2t^2}{n} - \frac{1}{2n^3}\right)\sqrt{4n^2t^2+1}$$

and

$$m_4(K_n^*, t) = 8t^4 - \frac{4t^3}{n} - \frac{2t}{n^3} + \frac{1}{2n^4} + \left(-\frac{4t^3}{n} + \frac{2t^2}{n^2} + \frac{2t}{n^3} - \frac{1}{2n^4}\right)\sqrt{4n^2t^2+1}.$$

So if we take $\alpha = 1$ we have for $t > 0$

$$\lim_{n\to+\infty} nm_1(K_n^*, t) = -\frac{1}{2} = \ell_1(t)$$

while for $t = 0$, $m_1(K_n^*, 0) = 0$, then $\ell_1(0) = 0$. For what concerns $\ell_2(t)$ we have that $\ell_2(t) = t$ for $t \ge 0$. For condition (3) it is sufficient to check the second part. We have

$$\sum_{\left|\frac{k}{n}-t\right|\ge\delta} \mathrm{e}^{-nu_n(t)} \frac{(nu_n(t))^k}{k!}\left(\frac{k}{n}-t\right)^2 \le \frac{1}{\delta^2} \sum_{\left|\frac{k}{n}-t\right|\ge\delta} \mathrm{e}^{-nu_n(t)} \frac{(nu_n(t))^k}{k!}\left(\frac{k}{n}-t\right)^4$$
$$\le \frac{1}{\delta^2} m_4(K_n^*, t)$$

and applying Lemma 4.1 in [22], for which $m_4(K_n^*, t) = \mathcal{O}(n^{-2})$, the condition holds.

So we have the following Voronovskaja formula obtained in [22]

$$\lim_{n\to+\infty} n[(L_n^*f)(t) - f(t)] = -\frac{1}{2}f'(t) + t\frac{f''(t)}{2},$$

at every point $t > 0$ in which $f''(t)$ exists.

Applying Theorem 3 we have

Corollary 4. *Let $f \in C^2$ be fixed and let $t > 0$. Then there holds*

$$\left| n[(L_n^*f)(t) - f(t)] + \frac{1}{2}f'(t) - t\frac{f''(t)}{2} \right| \le |f'(t)|\mathcal{O}(n^{-1}) + \frac{|f''(t)|}{2}\mathcal{O}(n^{-1}) + \mathcal{O}(1)\widetilde{\omega}\left(f'', \mathcal{O}(n^{-1/2})\right).$$

Proof. Using the same calculations given in Lemma 3, it is sufficient to prove that

$$\left|nm_1(K_n^*, t) + \frac{1}{2}\right| = \mathcal{O}(n^{-1}), \qquad |nm_2(K_n^*, t) - t| = \mathcal{O}(n^{-1}).$$

Indeed we have

$$nm_1(K_n^*, t) + \frac{1}{2} = \frac{\sqrt{4n^2t^2+1} - 2nt}{2} = \frac{1}{2(\sqrt{4n^2t^2+1} + 2nt)} = \mathcal{O}(n^{-1})$$

and analogously for the second relation. □

Remark 5. Note that a similar King version of the operator studied in Section 3, Example 2, can be obtained by choosing

$$u_n(t) = \frac{-1 + \sqrt{1 + 4(n+\gamma)^2t^2}}{2n}.$$

In this way we obtain the same Voronovskaja formula.

As before, from Corollary 4, the convergence in the Voronovskaja formulae is uniform on every compact interval in $]0, +\infty[$.

5. Generalized sampling operators

The theory developed can also be applied to generalized sampling operators (see [28]). We will illustrate a new special case namely the generalized Jackson operators with kernel

$$J_{\gamma,\beta}(t) = c_{\gamma,\beta}\mathrm{sinc}^{2\beta}\left(\frac{t}{2\gamma\beta\pi}\right),$$

with $t \in I = \mathbb{R}$, $\beta \in \mathbb{N}$, $\gamma \geq 1$, $c_{\gamma,\beta}$ is a normalization constant and $\mathrm{sinc}u := \frac{\sin\pi u}{\pi u}$. It is well known (see [24,25]) that $J_{\gamma,\beta}$ is bandlimited to the interval $[-1/\gamma, 1/\gamma]$. In this instance our operator takes the form

$$(S_n^{\gamma,\beta}f)(t) = \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(nt-k)f\left(\frac{k}{n}\right), \quad n \in \mathbb{N},\ t \in I = \mathbb{R}.$$

Here $\nu_{n,k} = \frac{k}{n}$, for $k \in \mathbb{Z}$ and the kernel $K_n(t, \nu_{n,k})$ is defined by

$$K_n\left(t, \frac{k}{n}\right) = J_{\gamma,\beta}\left(n\left(t - \frac{k}{n}\right)\right) = J_{\gamma,\beta}(nt-k).$$

We put for $j \in \mathbb{N}_0$

$$\widetilde{m}_j(J_{\gamma,\beta}, t) := \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(t-k)(k-t)^j$$

and

$$\widetilde{M}_j(J_{\gamma,\beta}, t) := \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(t-k)|k-t|^j.$$

Then we have

$$m_j(K_n, t) = \frac{1}{n^j}\widetilde{m}_j(J_{\gamma,\beta}, nt), \qquad M_j(K_n, t) = \frac{1}{n^j}\widetilde{M}_j(J_{\gamma,\beta}, nt).$$

Since $J_{\gamma,\beta}$ is bandlimited to $[-1/\gamma, 1/\gamma]$ and using the Poisson summation formula in the form (see [11])

$$(-\mathrm{i})^j \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(t-k)(t-k)^j \sim \sum_{k=-\infty}^{+\infty} \widehat{J}^{(j)}_{\gamma,\beta}(2\pi k)\mathrm{e}^{\mathrm{i}2\pi kt}$$

we have that

$$(-\mathrm{i})^j \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(t-k)(t-k)^j = \widehat{J}^{(j)}_{\gamma,\beta}(0).$$

Here the Fourier transform of a function $g \in L^1(\mathbb{R})$ is defined by

$$\widehat{g}(v) = \int_{-\infty}^{+\infty} g(u)\mathrm{e}^{-\mathrm{i}vu}\mathrm{d}u.$$

Now it is easy to see that $\widehat{J}'_{\gamma,\beta}(0) = 0$ while

$$\widehat{J}''_{\gamma,\beta}(0) = -\int_{-\infty}^{+\infty} t^2 J_{\gamma,\beta}(t)\mathrm{d}t =: -A_{\gamma,\beta}.$$

So for every $t \in \mathbb{R}$

$$\widetilde{m}_1(J_{\gamma,\beta}, t) = 0, \qquad \widetilde{m}_2(J_{\gamma,\beta}, t) = A_{\gamma,\beta} > 0.$$

As a consequence we obtain for $\alpha = 2$, $\ell_1(t) = 0$ and

$$\lim_{n\to+\infty} n^2 m_2(K_n, t) = A_{\gamma,\beta} = \ell_2(t).$$

Moreover if $\beta \geq 3$ then (see [25, Remark 3.2(d)])

$$\sup_{t\in\mathbb{R}} \widetilde{M}_{2\beta-2}(J_{\gamma,\beta}, t) < +\infty,$$

hence we have

$$M_3(K_n, t) = \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(nt-k)\left|t - \frac{k}{n}\right|^3 = \frac{1}{n^3}\widetilde{M}_3(J_{\gamma,\beta}, nt) = \mathcal{O}(n^{-3})$$

uniformly with respect to $t \in \mathbb{R}$. Analogously $M_2(K_n, t) = \mathcal{O}(n^{-2})$ uniformly with respect to $t \in \mathbb{R}$.

In order to prove assumption (3), for $\beta \geq 3$ it is sufficient to show that for every $\delta > 0$,

$$\sum_{|\frac{k}{n}-t|\geq\delta} J_{\gamma,\beta}(nt-k)\left(\frac{k}{n} - t\right)^2 = o(n^{-2}) \quad (n \to +\infty)$$

for $t \in \mathbb{R}$. Firstly note that for every $R > 0$

$$\sum_{|k-t|\geq R} J_{\gamma,\beta}(t-k)(t-k)^2 \leq \frac{1}{R^2}\sup_{t\in\mathbb{R}} \widetilde{M}_4(J_{\gamma,\beta}, t).$$

This implies that

$$\lim_{R\to+\infty} \sum_{|k-t|\geq R} J_{\gamma,\beta}(t-k)(t-k)^2 = 0$$

uniformly with respect to $t \in \mathbb{R}$. Hence for a given $\delta > 0$ and $n \in \mathbb{N}$ such that $n\delta > R$

$$\sum_{|\frac{k}{n}-t|\geq\delta} J_{\gamma,\beta}(nt-k)\left(\frac{k}{n} - t\right)^2 \leq \frac{1}{n^2}\sum_{|k-nt|\geq n\delta} J_{\gamma,\beta}(nt-k)(k-nt)^2$$

and so the assertion follows.

As a consequence we obtain the following Voronovskaja formula for the generalized Jackson operator

$$\lim_{n\to+\infty} n^2[(S_n^{\gamma,\beta} f)(t) - f(t)] = A_{\gamma,\beta}\frac{f''(t)}{2},$$

at every point $t \in \mathbb{R}$ in which $f''(t)$ exists.

For what concerns the quantitative estimate we get

Corollary 5. *Let $f \in C^2$ be fixed and let $t \in \mathbb{R}$. Then, for $\beta \geq 3$ there holds*

$$\left|n^2[(S_n^{\gamma,\beta} f)(t) - f(t)] - A_{\gamma,\beta}\frac{f''(t)}{2}\right| \leq \mathcal{O}(1)\widetilde{\omega}\left(f'', \frac{1}{3n}\frac{\sup_{t\in\mathbb{R}} \widetilde{M}_3(J_{\gamma,\beta}, t)}{A_{\gamma,\beta}}\right).$$

Proof. It is sufficient to remark that

$$\frac{M_3(K_n, t)}{M_2(K_n, t)} = \frac{1}{n}\frac{\widetilde{M}_3(J_{\gamma,\beta}, nt)}{\widetilde{M}_2(J_{\gamma,\beta}, nt)} \leq \frac{1}{n}\frac{\sup_{t\in\mathbb{R}} \widetilde{M}_3(J_{\gamma,\beta}, t)}{A_{\gamma,\beta}} = \mathcal{O}(n^{-1}). \quad \square$$

As a consequence, the convergence in the Voronovskaja formula for the Jackson sampling operator is uniform in $\mathbb{R}$.

As to the calculation of the constant $A_{\gamma,\beta}$ we have

$$A_{\gamma,\beta} = -\widehat{J}''_{\gamma,\beta}(0) = 16\beta^3\gamma^3 \int_0^{+\infty} \frac{\sin^{2\beta} t}{t^{2\beta-2}}\mathrm{d}t.$$

For example for $\beta = 3$ we get

$$A_{\gamma,\beta} = 16\beta^3\gamma^3 = 432\gamma^3.$$

For other values of β one can use the formula 12 p. 454 in [29].

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